

# Gravitational Lagrangian Dynamics of Cold Matter Using the Deformation Tensor.

Edouard Audit & Jean-Michel Alimi

<sup>1</sup> Laboratoire d'Astrophysique Extragalactique et de Cosmologie, CNRS URA 173, Observatoire de Paris-Meudon, Meudon 92195 CEDEX, France

**Abstract.** In this paper we present a new local Lagrangian approximation to the gravitational dynamics of cold matter. We describe the dynamics of a Lagrangian fluid element through only one quantity, the deformation tensor. We show that this tensor is clearly suited to the study of gravitational dynamics and, moreover, that knowing its evolution is enough to completely describe a fluid element. In order to determine this evolution, we make some physical approximations on the exact dynamical system and we thus obtain a closed local system of differential equations governing the evolution of the deformation tensor. Our approximate dynamics treat exactly the conservation of mass, of the velocity divergence and of the shear and is exact in the case of planar, cylindrical and spherical collapses. It also reproduces very accurately the evolution of all dynamical quantities for a very wide class of initial conditions as illustrated by a detailed comparison with the homogeneous ellipsoid model. Beside, we highlight, for the first time, the important dynamical role played by the Newtonian counterpart of the magnetic part of the Weyl tensor.

**Key words:** Cosmology: theory; dark matter; Large-scale structure of the Universe

## 1. Introduction

Two main analytical approaches have been used to study non-linear gravitational instability dynamics in cosmology. The first one is Eulerian, the fundamental quantities are then the density and velocity fields evaluated at the comoving Eulerian coordinate  $\mathbf{x}$  (Goroff et al. 1986, Grinstein & Wise 1987). The second method is that of Lagrangian trajectories (Zeldovich 1970, Moutarde et al. 1991, Buchert 1993, Bouchet et al. 1995). However, since the equations governing the motion of a self-gravitating dust are highly non-linear and non-local, finding an exact solution in the gen-

eral case, both in the Eulerian or Lagrangian approaches is impossible. Consequently, one is then forced either to restrict the problem to particular geometries (spherical or planar) or to use approximation techniques such as a perturbative theory. In the first case, one can study the very nonlinear evolution of a single object, but the range of initial conditions is then by definition very limited. The second case applies to any type of initial conditions, it consists of expanding the *solutions* of dynamical equations. In the Eulerian formalism one develops the density contrast and peculiar velocity fields as follow:

$$\delta = \sum_{n=0}^{n=\infty} \epsilon_{\delta}^{(n)} \delta^{(n)} \quad v = \sum_{n=0}^{n=\infty} \epsilon_v^{(n)} v^{(n)} \quad (1)$$

These developments are introduced in the dynamical equations (Mass conservation, Euler and Poisson equations). The equations are then solved keeping only terms in  $\epsilon$  and after in  $\epsilon^2$  and so on. The Eulerian expansion of the density contrast is valid when  $\delta < 1$ , but when  $\delta \approx 1$ , neglecting the term of high order is no longer possible. In the Lagrangian formalism the solution is obtained as a perturbed trajectory about the linear displacement

$$\mathbf{P}(\mathbf{q}, t) = \sum_{n=0}^{n=\infty} \epsilon_P^{(n)} \mathbf{P}^{(n)}(\mathbf{q}, t) \quad (2)$$

where  $\mathbf{q}$  is the Lagrangian coordinate. This kind of expansion allows one to follow the dynamics longer than in the Eulerian description, since the Lagrangian picture is intrinsically non-linear in the density field. The first order recovers the Zeldovich approximation (Zeldovich 1970). Higher orders provide corrections but the calculation rapidly becomes very tedious (Moutarde et al. 1991, Buchert 1992) and their domain of application are in fact restricted to the quasi-linear regime ( $\delta$  less than a few).

In this paper we propose a new Lagrangian approximation. It allows to compute the local properties of a fluid element during its motion, eventhough it does not determine its trajectory. From this point of view, our work is in the same spirit as that of Bertschinger & Jain (1994) ,

however, in our case, the nonlinear dynamic will be studied through only one quantity, the deformation tensor

$$\mathcal{J}_{ij} = \mathbf{1}_{ij} + \mathcal{W}_{ij} = \left( \frac{\partial x_i}{\partial q_j} \right). \quad (3)$$

$\mathbf{x}$  and  $\mathbf{q}$  are respectively the Eulerian and Lagrangian coordinates of a fluid element.

The deformation tensor when it is not singular, contains all the dynamical information on a fluid element excepted its trajectory. From this tensor, and from its time-derivatives, one can compute the density field, the velocity divergence, the shear tensor and the tidal tensor. Consequently, the influence of all these quantities which are very operative during the evolution of a fluid element (Bertschinger & Jain 1994, R. van de Weygaert & A. Babul 1994), will then be naturally taken into account in our formalism. Moreover as  $\mathcal{W}_{ij}$  are by definition always smaller than 1 (in absolute value), for a collapsing fluid element, they are very suitable quantities for all approximate perturbative description and can allow one to follow the evolution of large density contrasts or large displacements, this is not possible with the usual Eulerian or Lagrangian perturbative approach. If a fluid element is expanding in one direction the corresponding  $\mathcal{W}_{ij}$  can be greater than 1. In that respect, the deformation tensor is more adapted to describe collapsing regions than expanding voids.

We deduce the nonlinear dynamics of the deformation tensor by using an alternative procedure to the usual perturbative expansion of the solutions of the exact dynamical system. Our procedure consists, after some physical approximations, in deducing directly from the exact system a new approximate dynamical system where the non-locality of the gravitational dynamic introduced by the Poisson equation has been cancelled. The new closed set of local Lagrangian equations is then solved exactly. It takes into account nonlinear effects which could be absent in a perturbative solution. In other fields of research similar procedures have been used. It is the case for example, in the turbulent hydrodynamic or in the state solid physic, where some important physical processes, as the generation of solitons and the tunnel effect respectively, can not be explained by a perturbative solution. Our system reproduces exactly the spherical, cylindrical and planar collapses, but it also allows one to describe a much wider class of initial conditions and to follow the dynamics into the highly nonlinear regime.

A comparative analysis of our approximation with the results of Bertschinger & Jain 1994 and Lagrangian perturbative solutions has also been performed. It has allowed a better understanding of the respective influence of different dynamical quantities during the collapse, as for example the Newtonian limit of magnetic part of Weyl tensor which has been extensively discussed in recent papers (Bertschinger & Jain 1994,

Bertschinger & Hamilton 1994,

Kofman & Pogosyan 1995, Ellis & Dunsby 1995)

The outline of the paper is the following: In section 2 we present the dynamical equations of the deformation tensor and our approximations on these equations. In section 3 we give a physical explanation to our approximations and evaluate their accuracy. We also suggest a classification of collapses according to their geometrical properties. We compare our results with other approximate descriptions in section 4 Finally, conclusions and a summary of our results are given in section 5 The complete perturbative solutions to our dynamics is given in the appendix.

## 2. Dynamics of the Deformation Tensor

The evolution of the cold matter in an expanding Universe is governed by the following set of equations:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla_{\mathbf{x}} \cdot (1 + \delta) \mathbf{v} = 0 \quad (4)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a} (\mathbf{v} \cdot \nabla_{\mathbf{x}}) \mathbf{v} + \frac{\dot{a}}{a} \mathbf{v} = -\frac{1}{a} \nabla_{\mathbf{x}} \Phi \quad (5)$$

$$\nabla_{\mathbf{x}}^2 \Phi = 4\pi G a^2 \bar{\rho} \delta \quad (6)$$

$\Phi$  is the gravitational potential of the fluctuating part of the matter density, ( $\bar{\rho}$  is the average matter density,  $\delta$  is the density contrast)  $\mathbf{v}$  is the peculiar velocity ( $\mathbf{v} = a \frac{d\mathbf{x}}{dt}$ ) and  $a$  is the scale factor. The non-locality introduced by the Poisson equation (6) makes this system very difficult to integrate. However, by using the deformation tensor  $\mathcal{J}$ , it becomes possible to derive an approximate local description.

First, we change from the Eulerian to the Lagrangian time derivative and we introduce the conformal time  $\tau$  defined by  $d\tau = \frac{dt}{a^2}$ , the previous set of equations can then be written as:

$$\frac{d\delta}{d\tau} + a(1 + \delta)\theta = 0 \quad (7)$$

$$\frac{d^2 \mathbf{x}}{d\tau^2} = -\nabla_{\mathbf{x}} \tilde{\Phi} = a \frac{\partial \mathbf{v}}{\partial \tau} + \dot{a} \mathbf{v} + a^2 (\mathbf{v} \cdot \nabla) \mathbf{v} \quad (8)$$

$$\nabla_{\mathbf{x}}^2 \tilde{\Phi} = 4\pi G a^4 \bar{\rho} \delta \quad (9)$$

where  $\tilde{\Phi} = a^2 \Phi$  and  $\theta$  is the velocity divergence.

The conservation of mass (equation (7)) can be rewritten in terms of  $\mathcal{J}$  as  $1 + \delta = \frac{1}{J}$  where  $J = \det \mathcal{J}$ . The derivative with respect to  $\mathbf{q}$  of equation (8) and the properties of commutation between the operators  $\nabla_{\mathbf{q}}$  and  $\frac{d}{d\tau}$  yields

$$d_\tau^2 \mathcal{J} = -\mathcal{J} * \mathcal{F} \quad (10)$$

where  $*$  denotes a matrix product. Finally, by introducing  $\mathcal{F}_{ij} = \nabla_{x_i} \nabla_{x_j} \tilde{\Phi}$  in equation (9), the new system of equations becomes:

$$\begin{cases} 1 + \delta = \frac{1}{J} \\ d_\tau^2 \mathcal{J} = -\mathcal{J} * \mathcal{F} \\ Tr(\mathcal{F}) = \beta(\tau) \delta \end{cases} \quad (11)$$

We have defined  $\beta(\tau) = 4\pi G a^4(\tau) \bar{\rho}(\tau)$ . All the effects of  $\Omega$  on the dynamics are contained in this function  $\beta$ . In an  $\Omega = 1$  universe  $\beta = \frac{6}{\tau^2}$ . These equations were first written by Lachieze-Rey (Lachieze-Rey 1993) who proposes a solution where the deformation tensor has the particular form:  $\mathcal{J}(\mathbf{q}, t) = D_{\mathbf{q}}(t) \mathcal{J}_0(\mathbf{q})$ .  $\mathcal{J}$  is a tensor field which depends only on the Lagrangian coordinates and  $D_{\mathbf{q}}(t)$  is a scalar function of time which is given by a differential equation whose coefficients are functions of the eigenvalues of  $\mathcal{J}_0(\mathbf{q})$ . Assuming that the deformation tensor remains proportional to its initial value during the evolution of the fluid element greatly simplifies the dynamics. However, this is a very restrictive assumption, and such a solution admits a perturbative expansion which generally differs even at second order from the usual Lagrangian perturbative solution.

In the general case  $\mathcal{J}$  has nine independent components and system (11) cannot be solved. However, if the flow is irrotational then  $\mathcal{S}_{ij} = \frac{1}{a} \frac{\partial \mathbf{V}_i}{\partial \mathbf{x}_j}$  is symmetric and since  $\mathcal{S} = \frac{1}{a} \mathcal{J}^{-1} * d_\tau \mathcal{J}$ , this implies that  $\mathcal{J}$  is also symmetric, with only six independent components. In the rest of this paper we restrict ourself to irrotational flow. Moreover, if  $\mathcal{J}$  is symmetric, it can be diagonalized. We perform this diagonalization in the initial conditions and we consider that the first and second order time-derivative of non-diagonal components are initially vanishing, in order that  $\mathcal{J}$  remains diagonal during all the evolution. This assumption restricts somewhat the range of initial conditions. However, as it has been shown by Moutarde *et al.* (1991) in the case of the evolution of three caustics the non diagonal terms which appear during the evolution, do not modify strongly the dynamics of the diagonal terms.

From a diagonal deformation tensor  $\mathcal{J}_i = 1_i + w_i$  all the usual dynamical quantities can then be easily expressed

$$\delta = \frac{1}{J} - 1 = \frac{1}{(1+w_1)(1+w_2)(1+w_3)} - 1 \quad (12)$$

$$\theta = \sum_{i=1}^3 \frac{\dot{w}_i}{a(1+w_i)} \quad (13)$$

$$\sigma_i = \sigma_{ii} = \frac{\dot{w}_i}{a(1+w_i)} - \frac{1}{3}\theta \quad (14)$$

$$\mathcal{F}_i = \mathcal{F}_{ii} = -\frac{\ddot{w}_i}{(1+w_i)} \quad (15)$$

where  $\theta$  and  $\sigma$  are respectively the velocity divergence and the shear tensor. It is important to note that with the above expressions a major part of the dynamics is then already exactly described. The conservation-evolution equations of  $\delta$  (continuity equation), of  $\theta$ , (Raychaudhuri equation) and of  $\sigma$ , are in fact automatically satisfied. We only have to find the evolution equations for the  $w_i$  components. Such an equation will be derived from an local approximation of the non-local Poisson equation.

Equations (11) give:

$$Tr(\mathcal{F}) = -Tr(\mathcal{J}^{-1} d_\tau^2 \mathcal{J}) = \beta(\tau) \left( \frac{1}{J} - 1 \right) \quad (16)$$

In terms of  $w_i$  this equation can be written:

$$\begin{aligned} \sum_{i=1,3} \{ (1 + w_j + w_k + w_j w_k) d_\tau^2 w_i \\ = \beta(\tau) \left( 1 + \frac{w_j + w_k}{2} + \frac{w_j w_k}{3} \right) w_i \} \end{aligned} \quad (17)$$

where  $(i, j, k)$  is a circular permutation of  $(1, 2, 3)$ . This equation does not contain enough information in order to evaluate the evolution of each  $w_i$  component. However its re-writting, respectively, in one-dimensional, two-dimensional and three dimensional geometries suggest reasonable physical approximations, from which we will deduce an approximate evolution of the deformation tensor.

In a one-dimensional geometry, equation (17) reduces to:

$$d_\tau^2 w_1 = \beta(\tau) w_1 \Rightarrow w_1(q, t) = a(t) w_1^0(q) \quad (18)$$

where  $w_1^0(q)$  is the Lagrangian deformation field linearly extrapolated to  $a = 1$ . This solution is exact for a planar collapse. A similar evolution for the three components of  $\mathcal{W}$  tensor corresponds to the Zeldovich approximation.

In a two-dimensional geometry, equation (17) can be written, where  $(i, j)$  is a circular permutation of  $(1, 2)$

$$\sum_{i=1,2} \{ (1 + w_j) d_\tau^2 w_i = \beta(\tau) \left( 1 + \frac{w_j}{2} \right) w_i \} \quad (19)$$

In this equation  $w_1$  and  $w_2$  have a symmetric role. Splitting symmetrically this unique equation in two, we get:

$$(1 + w_j) d_\tau^2 w_i = \beta(\tau) (w_i + w_i w_j) + B_{ij} \quad (20)$$

In these equations  $B_{ij}$  components are unknown, however, they must obey the following constraints:

1.  $B_{ij} = -B_{ji}$  because the sum of equations (20) must reduce to equation (17).
2.  $w_j = 0$  implies  $B_{ij} = 0$  in order to be consistent with the one dimensional case.
3.  $w_i = w_j$  implies  $B_{ij} = 0$ . In this case we have only one unknown and equation (17) gives the solution.

Making the hypothesis that  $B_{ij}$  can be expressed as a polynomial of  $w_i$  the previous constraints gives:

- 3.  $\Rightarrow B_{ij} \propto (w_i - w_j)$
- 2.  $\Rightarrow B_{ij} \propto w_j (w_i - w_j)$
- 1.  $\Rightarrow B_{ij} \propto w_i w_j (w_i - w_j)$

This means that  $B_{ij}$  is at least of third order in  $w_i$ . Equation (20) with  $B_{ij} = 0$  is then exact up to second order included in  $w_i$  terms. It can therefore be used to compute the evolution of the deformation tensor with a very good accuracy.

Following the same procedure in three dimensional case we get the three following equations ( $i = 1, 2, 3$ ):

$$(1 + w_j + w_k + w_j w_k) d_\tau^2 w_i = \beta(\tau) \left(1 + \frac{w_j + w_k}{2} + \frac{w_j w_k}{3}\right) w_i + B_{ijk} \quad (21)$$

This time the constraints on  $B_{ijk}$  are:

1.  $B_{ijk} = B_{ikj}$ , because  $w_j$  and  $w_k$  play symmetrical roles towards  $w_i$ .
2.  $w_j = w_k = 0$  implies  $B_{ijk} = 0$  in order to be consistent with the one dimensional case.
3.  $w_i = w_j = w_k$  implies  $B_{ijk} = 0$ , because in this case, which corresponds to a spherical geometry, there is only one unknown and equation (17) then gives the exact solution.
4.  $w_k = 0$  implies  $B_{ijk} = B_{ij}$  in order to be consistent with the two dimensional case.
5.  $B_{ijk} + B_{jki} + B_{kij} = 0$ , because the sum of the equations (21) must reduce to equation (17).

Making again the hypothesis that  $B_{ijk}$  is a polynomial in terms of  $w_i$ , the polynomial of lowest order satisfying the constraints 1.) to 5.) is  $B_{ijk} = B_{ij} + B_{ik}$ .

Equations (21) with  $B_{ijk} = 0$

$$(1 + w_j + w_k + w_j w_k) d_\tau^2 w_i = \beta(\tau) \left(1 + \frac{w_j + w_k}{2} + \frac{w_j w_k}{3}\right) w_i \quad (22)$$

are then exact at least up to second order included in  $w_i$  terms. They form a close system of perturbed equations. It is important to notice that our approximation is both perturbative and geometrical. It can be considered as perturbative because we discard a term of at least third order in  $w_i$  in equation (21) and it can also be considered as a geometrical approximation because it is exact for planar, cylindrical and spherical collapse and only approximate in the general case. In the next section we discuss the physical meaning of the approximate dynamics of equations (22) and we show that our solution reproduces quite faithfully the nonlinear dynamic, in particular the evolution of density contrast.

### 3. Physical interpretation of the approximation

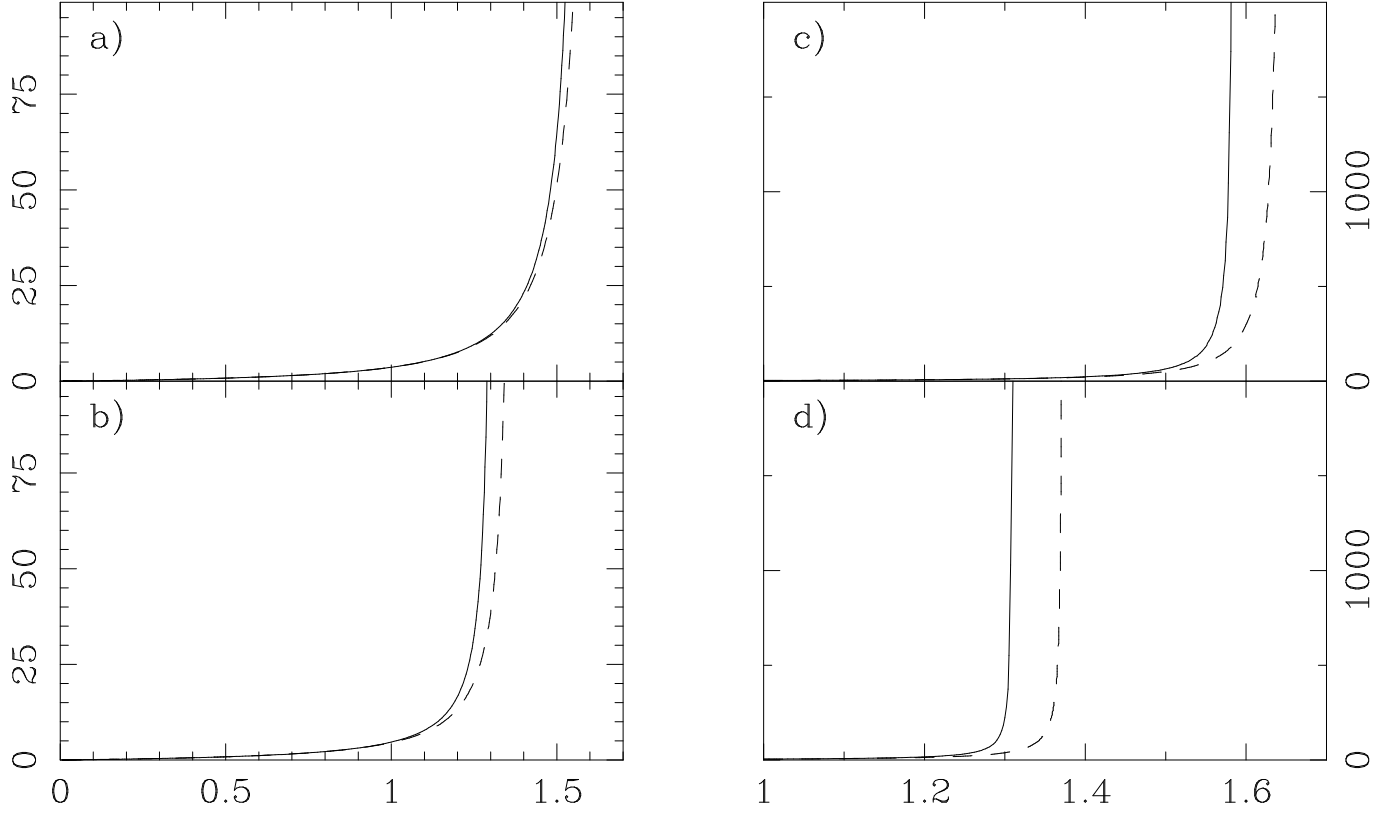
#### 3.1. Spherical and ellipsoidal collapses

The equations (22) which reduces respectively in one-dimensional and two-dimensional geometries to equations (18) and (20) (with  $B_{ij} = 0$ ), reproduces the exact planar and cylindrical dynamics. In spherical geometry, equations (22) also gives the exact solution. As a matter of fact, in such a case, there is only one free parameter, the density contrast  $\delta$  which is directly related to the unique component of the deformation tensor,  $w_{spherical} = (\frac{1}{1+\delta})^{1/3} - 1$ . Such a relation allows one then to identify the collapse of a homogeneous sphere with a density contrast  $\delta$  and a collapse in one point with a spherical environment defined by

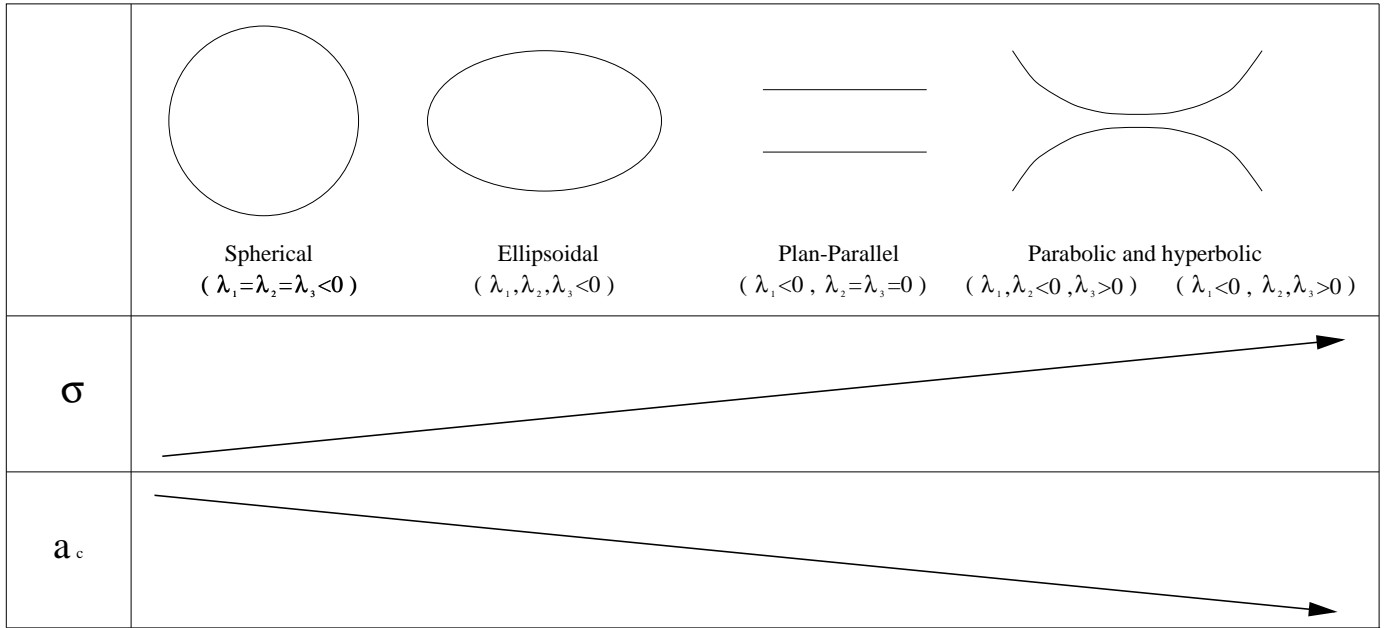
$w_{spherical}$  whose evolution is given by equations (22). In the following, in a similar way, we will identify finite scale collapses with local collapses with the same geometry.

When the three diagonal components of  $\mathcal{W}$  are different, it is natural to generalize the sphere to an ellipsoid. The equations (22) then describe the collapse of a point with an elliptic environment which means that the deformation tensor at such a point can be identified with the deformation tensor at the center of an homogeneous ellipsoid. The question is now to find such an ellipsoid. An ellipsoid of finite size is defined by three free parameters as, for example, the two axis ratios, and the initial density contrast (the local dynamical quantities at the center of the ellipsoid do not depend on the size of the ellipsoid but only on its shape). These three free parameters are not directly related to the initial eigen values of the deformation tensor. Consequently in order to link the ellipsoid of finite size and the initial deformation tensor we have used the tide tensor. This one is very meaningful for the dynamics and it is well defined in both cases. The tide tensor at the center of the ellipsoid can be computed using elliptic integrals, and it is related to  $\mathcal{W}$  by  $\mathcal{F} = -(\mathbf{1} + \mathcal{W}) d_\tau^2 \mathcal{W}$ . In Figures 1 we compare the evolution of the density contrast computed from equations (22) and for a homogeneous ellipsoid having the same initial tide tensor. In principle the ellipsoid generates inhomogeneities in the outside homogeneous background and these inhomogeneities in return perturb the density inside the ellipsoid. The dynamics of the homogeneous ellipsoid is computed by neglecting such a feedback effect which means that the density inside and outside the ellipsoid remains homogeneous. A detailed analysis of the ellipsoidal model can be found in V. Icke 1973, White S. & Silk J. 1979. In Figure 1, we see that our approximation reproduces very accurately the evolution of the density contrast at the beginning of the non-linear regime even for very non spherical cases (Figure 1b). The error is always of a few percent for  $\delta = 10$ . In the very non-linear phase (for  $\delta \approx 100$ ) the dynamical delay of our approximation is, in all cases, less than 5%. Figures 1c-1d show the final stage of the collapse. The collapse time when the density diverges, is estimated with an accuracy better than 5%. This one remains rather constant, whatever the ellipticity.

This generalization of the spherical collapse allows one to describe points with an anisotropic deformation tensor. The matter distribution around the fluid elements is approximated by an ellipsoid which can fit a generic matter distribution much better than a sphere. The picture of an ellipsoid is helpful in understanding the dynamics but one should not forget that we are evolving Lagrangian fluid elements and not a finite sized ellipsoid. Another important point is that for an elliptic collapse the tide tensor remains diagonal, this was our first approximation to obtain equations (22). Consequently, if we restrict ourself to this kind of collapse our only approximation is  $B_{ijk} = 0$



**Fig. 1.** This figure shows the evolution of the density contrast for two different ellipsoids. The y-axis is labeled in terms of the density contrast and the x-axis in terms of  $\tilde{a} = a * \delta_i / a_i$  where  $a$  is the scale factor and  $\delta_i$  and  $a_i$  are the initial density contrast and scale factor. The axis ratios of the ellipsoids are 1.24-0.8 for a) and c) and 2.21-0.45 for b), and d)



**Fig. 2.** This figure shows the geometrical classification of the different possible collapses.  $\sigma$  represents the eigenvalue of the shear on the fastest collapsing direction and  $a_c$  the collapse time of the fluid element. (The initial density contrast is supposed to be positive and equal in all cases).

and we have shown that discarding this term was a very good approximation.

### 3.2. Others Geometries

The evolution described by equations (22) is not restricted to the Lagrangian fluid elements with an elliptical environment for which the three diagonal components of  $\mathcal{W}$  are of the same sign (e.g. the opposite sign of  $\delta$ ). It can be applied to a wider class of initial conditions corresponding to an initial  $\mathcal{W}$  with eigenvalues of different sign. We call parabolic (resp. hyperbolic) the points which have an initial deformation tensor with two negative (resp. positive) and one positive (resp. negative) eigenvalues. For these points which correspond to more complex geometries it is not possible to find a simple distribution of matter with which they could be compared. They might be viewed as ellipsoids subject to a strong external tidal fields caused by surrounding matter. If the tidal field is strong enough to make one direction expand instead of collapsing the global matter distribution is not well describe by an ellipsoid. The parabolic and hyperbolic geometries allows to describe such cases.

The transition from elliptic to parabolic collapse is the planar (*i.e.* Zeldovich) collapse. As illustrated in Figure 2, the shear on the fastest collapsing direction increases and the collapse time decreases when one goes from elliptic to parabolic and finally to hyperbolic points. Parabolic and hyperbolic points collapse even if they are initially underdense due to the effect of the shear and of the tide. This illustrates that these two quantities, and not only the density contrast, play a key role in the dynamics. The only points which do not reach infinite density are elliptic points which are initially underdense or equivalently points with three positive eigenvalues for their initial deformation tensor. As in the case of the Zeldovich approximation, all the points with one initial negative eigenvalue for their deformation tensor, collapse. These points represent 92% of all points. This ratio can be computed by integrating the distribution function of the deformation tensor eigenvalues obtained by Doroshkevich (Doroshkevich 1970). Excepting cylindrical and spherical points which are highly symmetrical, the direction with the most negative initial eigenvalues of  $\mathcal{W}$  collapses first and alone. Consequently the collapsed objects will have a pancake shape.

In the next section we concentrate on elliptic collapses and make a detailed comparison of different approximations using not only the density contrast but also the shear and the tide tensor.

## 4. Comparison of different approximations:

In the previous section we have seen that our approximation reproduces with a good accuracy the evolution of the density contrast at the center of a homogeneous el-

lipsoïd (for fluid elements with an elliptic geometry). We will now push further this analysis in two directions. First, we will compare our approximation (hereafter VB as Vanishing  $B_{ijk}$ ), the one proposed by Bertschinger and Jain (Bertschinger & Jain 1994) (hereafter VH as Vanishing  $\mathcal{H}_{ij}$ ) and the first and second order developments (these developments are obtained by the method presented in the Appendix) to the homogeneous ellipsoid model. Since all these approximations keep the deformation tensor diagonal, it is natural to compare them with the dynamics of the homogeneous ellipsoid. Secondly, we will show that the density contrast alone can be misleading in determining the exactitude of a given dynamic. Consequently we will also study the evolution of the shear and tide tensor which, as we have seen, have a very important dynamical role and give valuable information about the geometry of the collapse.

To make the discussion more explicit we describe briefly the approximation suggested by Bertschinger and Jain. These authors have also proposed a closed local set of Lagrangian equations to describe the evolution of a fluid element. This set of equations for the irrotationnal case and for diagonal shear and tide tensor can be written as:

$$\dot{\delta} + a(1 + \delta)\theta = 0 \quad (23)$$

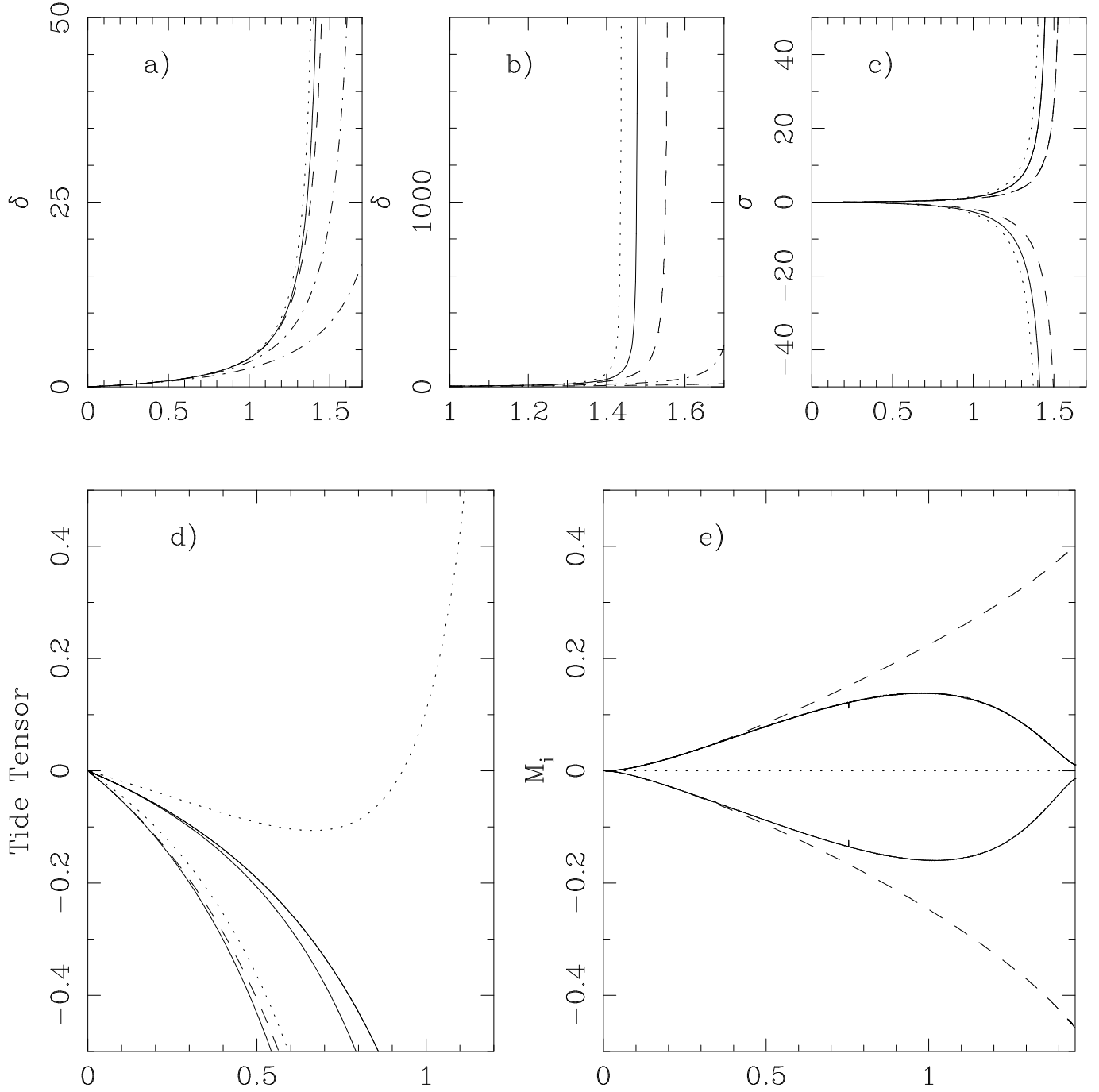
$$\dot{\theta} + \frac{\dot{a}}{a}\theta + \frac{a}{3}\theta^2 + a\sigma^i\sigma_i = -4\pi G\bar{\rho}a^3\delta \quad (24)$$

$$\dot{\sigma}_i + \frac{\dot{a}}{a}\sigma_i + \frac{2a}{3}\theta\sigma_i + a\sigma_i^2 - \frac{a}{3}(\sigma^k\sigma_k) = -a\mathcal{E}_i \quad (25)$$

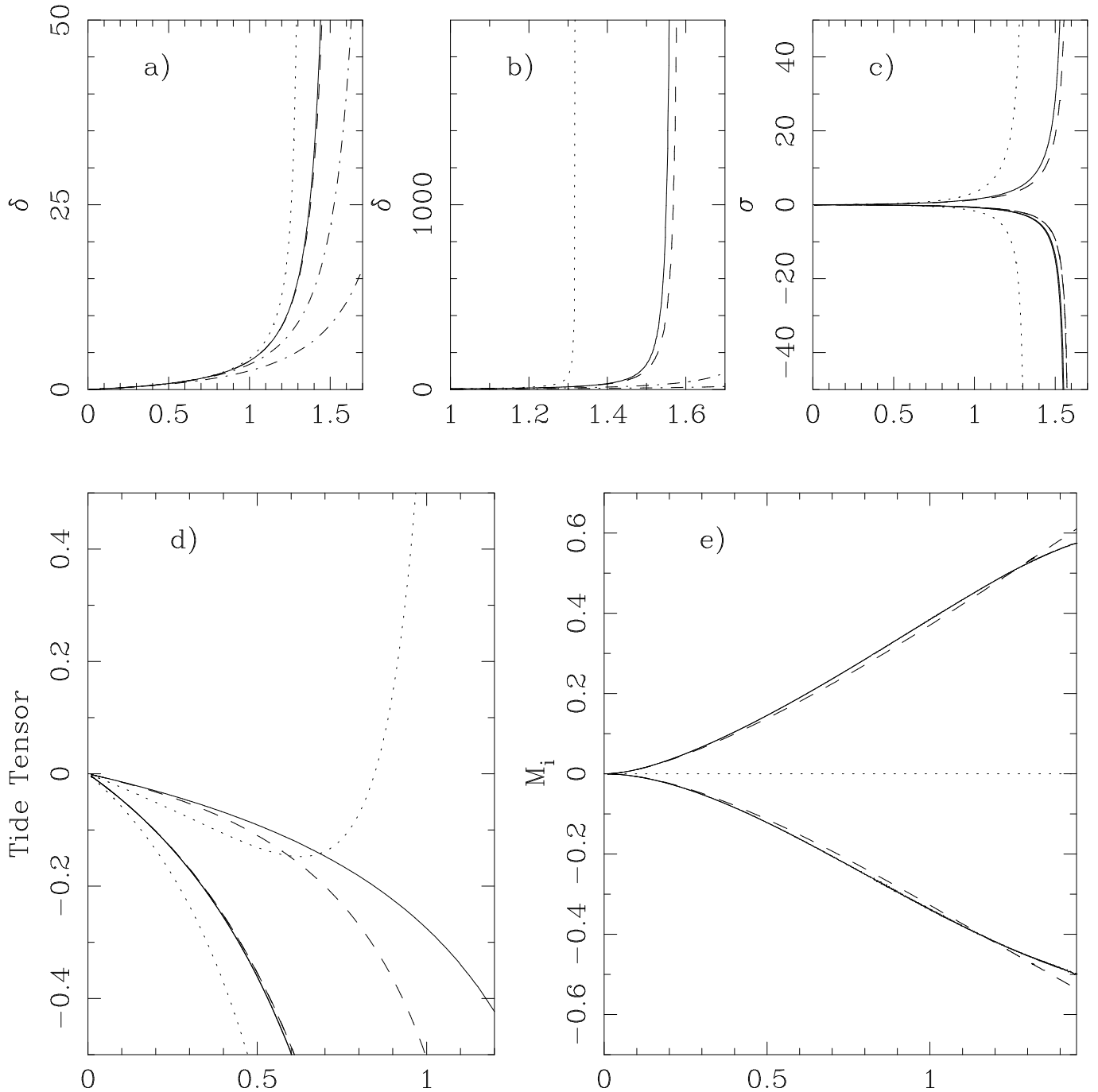
$$\dot{\mathcal{E}}_i - \frac{\dot{a}}{a}\mathcal{E}_i + \theta\mathcal{E}_i + \sigma^\parallel\mathcal{E}_\parallel - \sigma^\perp\mathcal{E}_\perp = -\Delta\pi\mathcal{G}\rho^{-1}\nabla\sigma_i \quad (26)$$

where  $\mathcal{E}_{ij}$  is the traceless part of the tide tensor  $\mathcal{F}_{ij} = \nabla_{x_i}\nabla_{x_j}\tilde{\Phi}$  ( $\mathcal{E}_{ij} = \mathcal{F}_{ij} - \frac{\delta_{ij}}{3}\nabla^2\tilde{\Phi}$ ) and the dot denote derivation with respect to  $\tau$ . Equations (23)-(25) are easily deduced from (7)-(9). Equation (26) governing the evolution of the tide tensor is obtained in the Newtonian limit, from general relativity by neglecting the magnetic part ( $\mathcal{H}_{ij}$ ) of the Weyl tensor (we will hereafter call  $M_i$  the term discarded in order to obtain equation (26) divided by the sum in absolute value of all the terms of this equation). The Newtonian expression of the Weyl tensor has been abundantly discussed recently (Kofman & Pogosyan 1995, Bertschinger & Hamilton 1994) but its dynamical influence remains unclear. During our analysis we will give a partial answer to this question.

Let us first examine the case of an ellipsoid which collapses first in one direction, the two other directions being equal. The ellipsoid is like a thick disc which collapse along its axis of symmetry. The evolution of the density contrast is perfectly described both by VB and VH even in the non-linear phase ( $\delta \approx 50$ ) (Fig. 3 a)). The improvement compared to first and second order developments is significant, which illustrates that integrating perturbed equations is more accurate than finding a perturbative solution. At the



**Fig. 3.** This figure displays the evolution of different dynamical quantities for the homogeneous ellipsoid model (full line), for VB (dashed line), for VH (dotted line) and for first and second order perturbative solutions (dashed-dotted line). The x-axis is always labeled in terms of  $\tilde{a}$  (cf. fig 2). Graph a) and b) shows the evolution of the density contrast with different scales, graph c) and d) shows the evolution of the eigenvalues of the shear and tide tensor and graph e) the evolution of  $M_i$  as defined in the text. The initial axis ratio of the ellipsoid are: 1-0.58.



**Fig. 4.** This figure displays the same graphs as in figure 3, but for an ellipsoid with an initial axis ratio: 2.4-1

collapse both VH and VB have a dynamical delay of a few percent (Fig 3 b)). VH under-estimates the collapse time and is slightly more accurate than VB which on the contrary over-estimates the collapse time. The analysis of the shear components gives similar conclusions. The shear is well described at the beginning of the non-linear phase and has a small dynamical delay at collapse (Fig 3 c)). How-

ever, the evolution of the tide tensor gives quite a different point of view (Fig 3 d)). First, we see that VH does not reproduce correctly the evolution of  $\mathcal{E}_{ij}$  even at a very early stage. In the linear regime  $\Phi \propto a^2$  which implies  $\dot{\mathcal{E}}_{ij} = 0$ . Thus, the first non-vanishing terms in  $\dot{\mathcal{E}}_{ij}$  are at least of second order. Consequently the dynamics of  $\mathcal{E}_{ij}$  is much affected by  $\mathcal{H}_{ij}$  which is generally not vanishing at second



order. On the other hand, in VB the term which is neglected is at least of third order it is therefore normal that the correct evolution for  $\mathcal{E}_{ij}$  is recovered at early stage. In the quasi-linear regime VB still gives quite a good description for the evolution of the tide tensor while VH does not reproduce the correct behavior. All these features can be understood by looking at the evolution of  $M_i$  (Fig 3 e))  $M_i$  grows very rapidly in the linear regime which translates immediately in a wrong evolution equation for the tide tensor for VH. VB gives the correct evolution of  $M_i$  in the linear phase. Near the collapse, since one direction collapses first, the fluid element evolves towards a planar geometry. For this reason  $M_i$  is again vanishing (compared to other quantities) at the collapse. VB is unable to follow this behavior and largely over-estimates the amplitude of  $M_i$  at the collapse.

Now we will turn our attention to the case where two directions are equal and collapse first. The ellipsoid then has a cigar shape and collapses in the two directions perpendicular to its axis of symmetry. For this particular geometry VB is extremely close to the exact (ellipsoidal) dynamics. The density contrast and the shear tensor are computed to a high accuracy (Fig 4 a), b) and c)). For both these quantities the dynamical delay at the collapse is less than 1%. VB is excellent in this case because the initial ellipsoid has a cigar shape and its dynamics is close to that of a cylinder (VB is exact for a cylindrical collapse). Looking now at the tide tensor we see that the two directions which collapse first are perfectly described while the third one, which should be zero for a cylinder, is not very well followed. VH does not give accurate results in this case. The fluid element tends, at collapse, towards a cylindrical geometry for which the Weyl tensor is not vanishing (contrary to what said by Bertschinger and Jain in their abstract). Consequently,  $M_i$  is playing an increasing dynamical role from the linear phase to the collapse which makes equation (26) a bad approximation. VB is extremely accurate for this particular ellipsoid because its dynamics is very close to that of a cylinder and we have shown that VB was exact in cylindrical geometry.

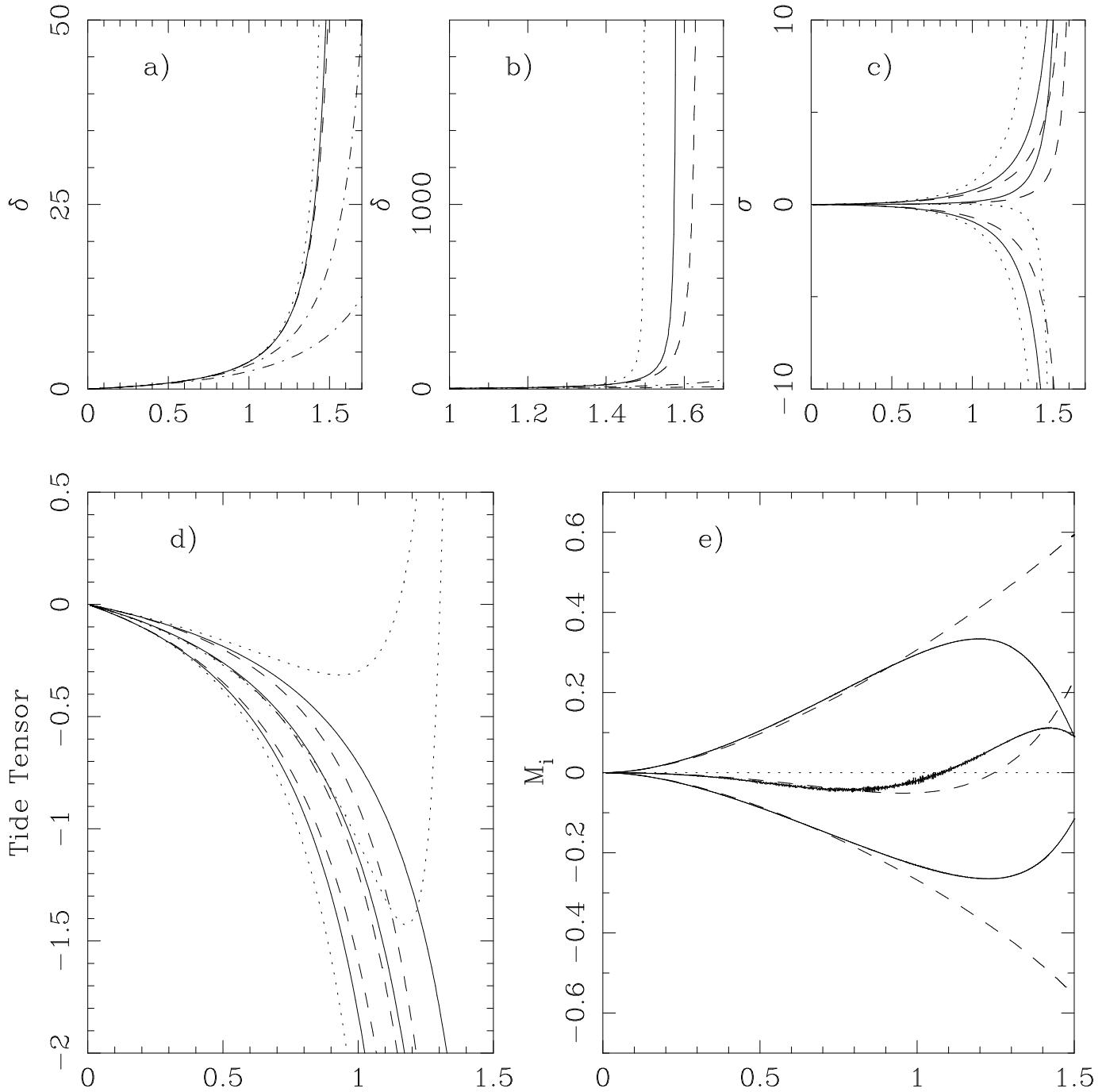
The two cases that we have studied previously were highly symmetrical. They were instructive in order to understand the role of the different dynamical quantities in each of the approximations. However, they are not representative of generic cosmological fluid elements. Consequently, we will now study the collapse of an “ordinary” ellipsoid with three different axes. We have plotted in figure 5 the evolution of only one such ellipsoid but the description we give is general and is the result of the analysis of many different cases. For these geometries both VH and VB describe the evolution of the density field very accurately (Fig 5 a) and b)). VB is a little more accurate in the non-linear phase and significantly better for the estimation of the collapse time. The error on the collapse time obtained from VB is always less than 5%, while it can be much greater for VH. VB always overestimates the

collapse time while it is underestimated by VH. The improvement compared to the second order perturbative solution is in all cases very important. This can be explained by the fact that a perturbative solution of equation (21) only preserves the perturbative aspect of our approximation but loses its geometrical properties which are “coded” in the equations. A perturbative solution is inexact for the cylindrical and the spherical collapse. The quality of the description of the shear and of the tide given by VB is quite similar to that of the density contrast (Fig 5c) and 5d)). Both these quantities are very well followed at the beginning of the non-linear phase and have a small dynamical delay at the collapse (this delay is a little greater for the tide). Even though our approximation reproduces quite faithfully the dynamical evolution of all the usual dynamical quantities, it is less accurate for  $M_i$ . In the first phase of the collapse the three components  $M_i$  start to grow, this phase is very well followed by VB. However, in a second phase, the  $M_i$  components return to zero under the correct dynamics (ellipsoidal model) while they continue to grow in VB (Fig 5 e)). As was mentioned before, when one direction collapses first the ellipsoid approaches a pancake shape when reaching the collapse and this explains why  $M_i$  evolve towards zero. However, even if they tend to zero at the collapse the  $M_i$  terms have a very important dynamical role. Even though  $M_i$  terms are small (but not vanishing) in the linear regime, they increase very rapidly to become one of the dominant terms in the equations (26). Consequently, VH rapidly becomes a very bad approximation. This explains the rather strange behavior of the tide tensor predicted by VH (Fig 5 d)). The error initially present in equation (26) propagates to equation (25) and then to (24) and finally to (23). One component of the shear is much affected by this error (Fig 5 c)), VH predicts two negative components for the shear while there is only one. The density contrast is less sensitive to it.

From the analysis of the different approximations we can conclude that the  $M_i$  terms slow down the collapse and couple the directions. VB which overestimates the amplitude of  $M_i$  always collapses too slowly and the three components of its tide tensor are much closer than in the exact ellipsoidal dynamics (Fig 5 d)). On the contrary, VH which completely discards  $M_i$ , collapses too quickly. Figures 5 c) and d) illustrate the importance of  $M_i$  for the dynamics and for the geometry of the collapsed objects.

## 5. Conclusion

In this paper we have presented and tested a new local Lagrangian approximation to the dynamics of a self-gravitating cosmological fluid. One originality of our approach was to describe the gravitational dynamics through only one quantity, the deformation tensor. We have shown that with this tensor it is possible to write the dynamical equations in a synthetic form and to highlight the role of different important dynamical quantities, especially the



**Fig. 5.** This figure displays the same graphs as in figure 3, but for an ellipsoid with an initial axis ratio: 1.24-0.8

shear and the tide. The evolution of the deformation is evaluated by deriving a close set of local equations from the exact dynamical system after two approximations. The first one supposes that the deformation tensor remains diagonal during its evolution, the second one consists in discarding terms of third order in terms of the deformation tensor component. The *approximate dynamics* thus

obtained, which reproduce exactly the planar, cylindrical and spherical collapses is then exactly integrated. Such a procedure is completely different from the method which consists in deriving a *perturbative solution* from the exact dynamics.

In order to test our approximation we have compared it with the collapse of a homogeneous ellipsoid and with

the approximation proposed by Bertschinger & Jain. From this analysis we conclude that our approximate dynamics is a very accurate tool for following all the dynamical quantities during the collapse. We have also, for the first time, shed some light on the important dynamical role played by the Newtonian counterpart of the magnetic part of the Weyl tensor, discarded by Bertschinger & Jain. This tensor slows down the collapse and couples the directions, discarding it reduces the collapse time significantly and greatly modifies the geometry of the collapse. The main weakness of our approximation is that, unlike the Zeldovich approximation, it is unable to reconstruct the trajectories. However many cosmologically interesting quantities, such as for example the mass function, can be computed without reconstructing the final Eulerian field. Furthermore one of the main interests of this work might be that it may give a better understanding of the very complex phenomena which occur in the non-linear phase of the gravitational dynamics.

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## 6. Appendix: Link with perturbative theory.

In section 4 we have seen that integrating our system is more accurate than finding a perturbative solution, at first and second order, of equations (22). Would this still be true for higher order solutions and does the perturbative solution of  $n^{th}$  order converge toward the exactly integrated solution of equation (22)? In order to answer these questions we have developed an analytical recursive formula which relates the perturbative solution of equation (22) at order  $n$  to the solution at order  $(n - 1)$ . The convergence of this development and the ability of such a perturbative solution to reproduce all the features of the exactly integrated solution will be the object of a forthcoming paper.

In this section we give a recursive formula to compute the solution of our system at any order. Of course this development will differ from the development of the exact dynamics beyond second order, but by going to higher orders we can analytically recover the accuracy of the numerically integrated solution.

Equations (21) can be written, after some simple algebra, as:

$$(1 + w_j + w_k + w_j w_k)(d_\tau^2 w_i - \beta(\tau) w_i) = -\beta(\tau) \left( \frac{w_j + w_k}{2} + \frac{2}{3} w_j w_k \right) w_i \quad (27)$$

It is possible to find a perturbative solution **at any order** of the preceding equation. Let us first introduce a few useful notations:

$$w_i = \sum_{n=1}^{\infty} w_i^{(n)}$$

$$\begin{aligned} \alpha_{ij}^{(n)} &= \sum_{l=1}^{n-1} w_i^{(l)} w_j^{(n-l)} \\ \beta^{(n)} &= \sum_{m=1}^{n-2} \sum_{l=1}^{n-m-1} w_i^{(m)} w_j^{(l)} w_i^{(n-l-m)} \\ P_i^{(n)} &= d_\tau^2 w_i^{(n)} - \beta(\tau) w_i^{(n)} \end{aligned}$$

With this notation it is possible to rewrite equation (27) as:

$$\begin{aligned} (1 + \sum_{n=1}^{\infty} (w_j^{(n)} + w_k^{(n)} + \alpha_j k^{(n)})) \sum_{n=1}^{\infty} P_i^{(n)} \\ = -\beta(\tau) \sum_{n=1}^{\infty} \left( \frac{\alpha_{ij}^{(n)} + \alpha_{ik}^{(n)}}{2} + \frac{2}{3} \beta^{(n)} \right) \quad (28) \end{aligned}$$

Keeping only terms of  $n^{th}$  order in this equation gives:

$$P_i^{(n)} = -\beta(\tau) \left( \frac{\alpha_{ij}^{(n)} + \alpha_{ik}^{(n)}}{2} + \frac{2}{3} \beta^{(n)} \right) + \sum_{l=1}^{n-2} (w_j^{(l)} + w_k^{(l)} + \alpha_{jk}^{(l)}) P_i^{(n-l)}$$

$P_i^{(n)}$  can be computed using only the  $w_j^{(k)}$  with  $k \leq n - 1$ . The differential equations  $d_\tau^2 w_i^{(n)} - \beta(\tau) w_i^{(n)} = P_i^{(n)}$  can be recursively integrated to any order. The fact that  $P_i^{(n)}$  is of order  $(n)$  (except  $P_i^{(1)}$  which is zero) is also shown recursively. If we keep only the linear growing mode at first order, these equations can be integrated very easily to a high order. Since equation (27) is exact only at second order the development will not be exact beyond that order.

However, as we have seen, the exact solution of equation (27) is much more accurate than the second order solution and by developing to a higher order we can have an analytical solution very close to the numerically-integrated one. Another interesting feature of this development is that since we can go to any order very easily it is possible to check if a given quantity can be accurately approximated by perturbative theory. It is possible that some quantities are not well described by perturbative theory and are only computable with the exact solution. For example  $d_\tau^2 w_i$  becomes infinite at collapse for the exact solution but any perturbative calculation will give a finite value.

## References

- Bertschinger, E. & Hamilton A.J.S. 1994, ApJ, **435**, 1
- Bertschinger, E. & Jain, B. 1994, ApJ, **431**, 486
- Bouchet, F., Colombi, S., Hivon, E. & Juszkiewicz, R. 1995, A&A, **296**, 575
- Buchert, T., 1992, MNRAS, **254** 729
- Buchert, T., 1993, A&A, **267** 51
- Ellis, G.F.R. & Dunsby, P.K.S., *Newtonian Evolution of the Weyl Tensor* 1995
- Goroff, M.H., Grinstein, B., Rey, S.-J., & Wise, M.B. 1986, ApJ, **311**, 6
- Grinstein, B. & Wise, M.B. 1987, ApJ, **320** 448

- Hui L.& Bertschinger E., astro-ph/9508114, preprint  
Kofman L. & Pogosyan D., 1995, ApJ **442**, 30  
Lachize-Rey, M., 1993, ApJ, **408**, 403L  
Moutarde, F., Alimi, J.-M., Bouchet, F.R., Pellat, R., and Rami, A., 1991, ApJ, **382**, 377  
Zeldovich, Ya.B. 1970, A.& A., **5**, 84  
van de Weygaert R. & Babul A. 1994, ApJ, **425**, L59  
White S.D.M. & Silk J. 1979, ApJ, **231**, 1  
Icke V. 1973 A&A, **27**, 1  
Doroshkevich A.G. 1970, Astrofizika **6**, 581